APPROXIMATION RESULTS FOR STOCHASTIC MULTIPLE-SETS SPLIT FEASIBILITY AND SPLIT EQUALITY PROBLEMS IN HILBERT SPACE

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Abstract

To handle real-life issues which are often entrenched in uncertainties, stochastic functional analysis has been developed. This paper has extended the convergence results of deterministic (classical) convex multiple-sets split feasibility and split equality to its stochastic equivalence. It has also proved that the solution of the stated optimization problems generated by random-typed iterative scheme converges almost surely and in quadratic mean, consequently in probability to random fixed-point in Hilbert space. These results extend, unify, and generalize different established deterministic results in the literature.

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INTRODUCTION

The stochastic multiple-sets split feasibility problem is to find

$$x(\omega) \in C = \bigcap_{i=1}^{p} C_i \text{ such that } Ax(\omega) \in Q = \bigcap_{j=1}^{q} Q_j \qquad (1.1)$$

while the stochastic multiple-sets equality problem is to find

$$x(\omega) \in C = \bigcap_{i=1}^{p} C_i \text{ and } y(\omega) \in Q = \bigcap_{j=1}^{q} Q_j \text{ such that } Ax(\omega) = By(\omega) \qquad (1.2)$$

where, $x(\omega)$ and $y(\omega)$ are random variables define on probability space $(\Omega, \Sigma, \mathrm{P})$, $\omega \in \Omega$, C *and* Q are respectively non-empty convex sets of two real Hilbert spaces H_1, H_2 and $A: H_1 \to H_2$ is a linear bounded operator with adjoint $A^*: H_2 \to H_1$.

Notations

Let $(\mathbf{\Omega}, \mathbf{\Sigma}, \mathbf{P})$ be a complete probability measure space and $(\boldsymbol{H}, \boldsymbol{B}(\boldsymbol{H}))$ be a measurable space with H as a separable Hilbert space, $\boldsymbol{B}(\boldsymbol{H})$ is a Borel sigma algebra of H, $(\mathbf{\Omega}, \mathbf{\Sigma})$ is a measurable space ($\mathbf{\Sigma}$-sigma algebra) and $\mathbf{P}$ is a probability measure on $\mathbf{\Sigma}$, that is, a measure with total measure one. A mapping $\boldsymbol{X}: \mathbf{\Omega} \to \boldsymbol{H}$ is called (a) H-valued random variable if $\mathbf{\Sigma}$ is $(\mathbf{\Sigma}, \boldsymbol{B}(\boldsymbol{H}))$-measurable, (b) strongly P-measurable if there exists a sequence $\{\boldsymbol{X}_n\}$ of P-simple functions converging to X P-almost surely except for some events with null probability. Due to the separability of a Hilbert space H, the sum of two H-valued random variables is an H-valued random variable. A mapping $\boldsymbol{T}: \mathbf{\Omega} \times \boldsymbol{H} \to \boldsymbol{H}$ is called a random operator if for each fixed $\boldsymbol{h} \in \boldsymbol{H}$, the mapping $\boldsymbol{T}(\_, \boldsymbol{h}): \mathbf{\Omega} \to \boldsymbol{H}$ is measurable.

The classical multiple-set split feasibility problem is considered deterministic and hence do not represent most real-life realities which are entrenched in uncertainties. The procedure for solution of deterministic multiple-sets split feasibility and split equality problems would have been a good substitute for its stochastic equivalence but for the reasons stated in (Udom and Nweke, 2021) which in summary are (i) stochastic functions are contaminated with noise (errors), hence they cannot not be fully specified, (ii) because of (i), the gradient of the stochastic function may not exist. This therefore gave rise to the study of random fixed-points theorem which are generalizations of classical fixed-points theorem applied to studies of stochastic feasibility problem which is a system influenced by random perturbation (i.e. a stochastic system). The purpose of this paper therefore is to extend the convergence results of deterministic multiple-sets split feasibility and split equality problems to its stochastic equivalence and prove that the solution of multiple-sets split feasibility and split equality problems generated by random-type iterative scheme with firmly non-expansive mapping converge in quadratic mean and/or almost surely, consequently in probability to random fixed-point in Hilbert space.

METHODOLOGY

Let y_n be an observation about an unknown parameter of a function $f(\cdot)$ at time n. Suppose that Z is the root of $f(\cdot): f(z)=0$. Let x_n be the estimate for Z at time n, then

$$y_{n+1} = f(x_n) + \varepsilon_{n+1} \tag{2.1}$$

where, ε_{n+1} is the observation error. By considering y_{n+1} observation on $f(\cdot)$ at n with the observation error ε_{n+1}, the problem has been reduced to seeking the root z of $f(\cdot)$ based on y_n.

To solve (2.1), Robbins and Monro (1951) have proposed a recursive algorithm known as RM algorithm which is given in (2.2) below as:

$$x_{n+1} = x_n - \alpha_n y_{n+1} \ , \ a_n > 0 \tag{2.2}$$

where, α_n is the step size.

For convergence of (2.2), using approximation algorithm with expanding truncations by applying Trajectory Subsequence method for the root set Γ of $f(\cdot): \Gamma = \{x \in \mathbf{R} : f(x) = 0\}$ with z being fixed in $\mathbf{R}^n$ and x_0 an arbitrary initial. Then x_n being the estimate at n which serves as the n^{th} approximation to Γ is given in the form:

$$x_{n+1} = (x_n - \alpha_n y_{n+1}) I_{[\|x_n - \alpha_n y_{n+1}\| \le M_{\sigma(n)}]} + x^* I_{[\|x_n - \alpha_n y_{n+1}\| > M_{\sigma(n)}]} \tag{2.3}$$

$$\text{where, } \sigma(n) = \sum_{i=1}^{n-1} I_{[\|x_i - \alpha_i y_{i+1}\| > M_{\sigma(i)}]}, \ \sigma(1) = 0 \tag{2.4}$$

$\mathbf{I}_{[inequality]}$ is an indicator function meaning that it equals 1 if the inequality indicated in the bracket holds or 0 otherwise, $\sigma(n)$ is the number of truncations up-to time n, $M_{\sigma(n)}$ serves as the truncation bound when the $(n+1)^{th}$ estimate is generated. If $\|x_n - \alpha_n y_{n+1}\| \le M_{\sigma(n)}$ is fulfilled, then (3.3) evolves as the RM algorithm otherwise the estimate at $n+1$ is pulled back to the pre-specified point x^*, then the truncation bound is enlarged from $M_{\sigma(n)}$ to $M_{\sigma(n+1)}$. $M_{\sigma(n)}$ is chosen to be be a sequence of positive numbers increasingly diverging to infinity while $x^*$is a fixed point in R^n.

Main Results

Theorem 3.1: *Let $(\Omega, \Sigma, \mathrm{P})$ be a complete probability measure space and H_1, H_2 two real finite dimensional Hilbert spaces. Let $C \subset H_1$ and $Q \subset H_2$ be two non-empty convex subsets of H_1 and H_2 respectively. Suppose that $P_C : C \to C$ and $P_Q : Q \to Q$ are two random projections onto C and Q respectively. Let $A : H_1 \to H_2$ be a bounded linear random operator with an adjoint $A^* : H_2 \to H_1$. Then the solution of the stochastic multiple-sets split feasibility problem (1.3) generated by $\{x_n(\omega)\}$ and defined by*

$$\begin{cases} x_0 = 1 \\ x_{n+1}(\omega) = (1-\beta_n) x_n(\omega) + \beta_n (P_C(\omega, x_n) - \lambda A^*(1-P_Q) A x_n(\omega)) , \\ \forall \ \beta_n \in (0,1) \text{ and } \lambda \in (0\ ,\ 2) \end{cases} \tag{3.1}$$

converges both almost surely and in quadratic mean to the random fixed-point $z(\omega) \in \Gamma$.

Theorem 3.2: *Let $(\Omega, \Sigma, \mathrm{P})$ be a complete probability measure space and H_1, H_2, H_3 three real finite dimensional Hilbert spaces. Suppose that $P_C : C \to C$ and $P_Q : Q \to Q$ are two random projections onto C and Q respectively where $C \subset H_1$ and $Q \subset H_2$ .Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear random operator with their respective adjoints $A^* : H_3 \to H_1$ and $B^* : H_3 \to H_2$. Then the solution of the stochastic multiple-sets split equality problem (1.4) generated by $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ and defined by*

$$\begin{cases} x_0 = 1 \\ y_n(\omega) = (1-\beta_n) x_n(\omega) + \beta_n \left(P_C(\omega, x_n) - \lambda_1 A^*(1-P_Q) A x_n(\omega)\right) \\ x_{n+1}(\omega) = (1-\alpha_n) x_n(\omega) + \alpha_n \left(P_C(\omega, y_n) - \lambda_2 B^*(1-P_Q) B y_n(\omega)\right) \\ \quad \forall \ \beta_n, \alpha_n \in (0,1), \lambda_1, \lambda_2 \in (0,2) \end{cases} \tag{3.2}$$

Converges in quadratic mean to the random fixed point $z(\omega)$

Remark: the proof of the above are available but not included here for space

NUMERICAL ILLUSTRATION

Let $H_1 = H_2 = \mathbf{R}^2$, $C = \{x \in \mathbf{R}^2 : x^2 + 5x - 50 - e_{1C} \le 0, 2x^2 + 88x + 80 - e_{2C} \le 0\}$

$and\, Q = \{x \in \mathbf{R}^2 : 2x^2 - 7x - 15 - e_{1Q} \le 0, 4x^2 + 16x + 15 - e_{2Q} \le 0\}$, where, $e_{iC} \sim \mathrm{N}(0, \sigma_{e_C}^2)$ and $e_{iQ} \sim \mathrm{N}$

Then the stochastic split feasibility problem is to find

$x \in C = x^2 + 5x - 50 - e_{1C} \cap 2x^2 + 88x + 80 - e_{2C}\ such\, that\ Ax \in Q = 2x^2 - 7x - 15 - e_{1Q} \cap 4x^2 + 16x$ -

Table 1: Numerical result of illustrative example

	$\alpha_n = \frac{1}{n+10}, \lambda = 1.5$, $\varepsilon =$ error bound for convergence criteria				
Iterations	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-7}$	$\varepsilon = 10^{-9}$	$\varepsilon = 10^{-11}$
1	-1.3852	-1.385242	-1.38524233	-1.3852423321	-1.385242332139
2	-0.2945	-0.294506	-0.29450617	-0.2945061705	-0.294506170537
3	-0.1418	-0.141832	-0.14183226	-0.1418322647	-0.141832264663
4	-0.1771	-0.177149	-0.17714872	-0.1771487204	-0.177148720368
5	-0.1705	-0.170532	-0.17053216	-0.1705321570	-0.170532157021
6	-0.1712	-0.171223	-0.17122294	-0.1712229385	-0.171222938452
7		-0.171195	-0.17119452	-0.1711945229	-0.171194522912
8		-0.171194	-0.17119404	-0.1711940431	-0.171194043121
9			-0.17119401	-0.1711940102	-0.171194010198
10				-0.1711940064	-0.171194006406
11				-0.1711940058	-0.171194005809
12				-0.1711940057	-0.171194005692
13					-0.171194005665
14					-0.171194005658
15					-0.171194005656

The result of the numerical illustrative shows that the solution of the multiple-sets split feasibility problem is -0.17. Convergence to this point was achieved using both in algorithm (3.1) and (3.2). It could be observed in both results that convergence is attained faster with lower error bound than with higher error bound.

CONCLUSION

This work has extended the convergence results for multiple-sets feasibility and split equality problem to its stochastic equivalent using random-type iterative algorithms. Convergence almost surely as well as in mean square (quadratic mean) and consequently in probability to random fixed-point in a Hilbert space were attained. Demonstration using numerical illustration shows applicability of the stochastic multiple-sets feasibility problem.