The piecewise power-law model

Nixon Jerez-Lillo[1, 4] Francisco Segovia[1, 4] Pedro Luis Ramos[1] Francisco Louzada Neto[2] Osafu Augustine Egbon[2, 3]

[1]Pontificia Universidad Católica de Chile, Chile.
[2]University of São Paulo, Brazil.
[3]Federal University of São Carlos, Brazil.
[4]National Agency for Research and Development (ANID), Scholarship Program, Doctorado Nacional.

Model definition

The survival function of the standard power-law distribution is given by

$$S(x) = \left(\frac{x}{x_0^*}\right)^{1-\alpha}, \quad x > x_0^*, \quad \alpha > 1.$$

Due to the simple form of the power-law, we easily extend the model for k change-points getting the piecewise power-law (pwplaw), which survival function can be represented by the following expression

$$S(x) = \sum_{i=1}^{k} \left(\frac{x}{x_{(i-1)}^*}\right)^{1-\alpha_i} C_{i-1} \cdot 1_{[x_{(i-1)}^*, x_{(i)}^*]}(x), \tag{1}$$

where

$$C_{i-1} = \prod_{j=1}^{i-1} \left(\frac{x_{(j)}^*}{x_{(j-1)}^*}\right)^{1-\alpha_j},$$

and $C_0 = 1$, $\alpha_i > 1$ and $x_{(i)}^* > x_{(i-1)}^*, \forall i = 1, \ldots, k$. From (1) the probability density distribution of the pwplaw model distribution is given by

$$f(x) = \sum_{i=1}^{k} \frac{\alpha_i - 1}{x_{(i-1)}^*} \left(\frac{x}{x_{(i-1)}^*}\right)^{-\alpha_i} C_{i-1} \cdot 1_{[x_{(i-1)}^*, x_{(i)}^*]}(x). \tag{2}$$

Figures 1 drawn different cases for the density and survival of the pwplaw distribution.

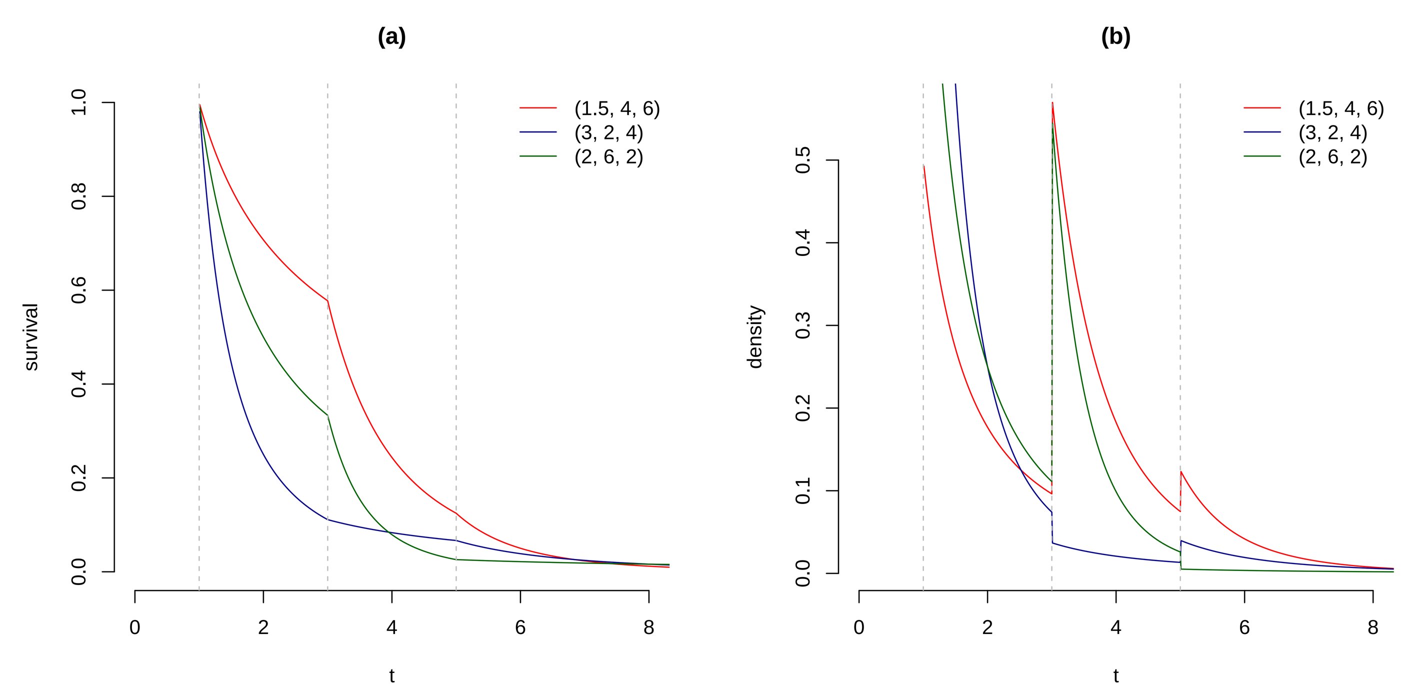

Figure 1. Survival (a), and density (b) functions with three changes points: $x_1^* = 1$, $x_2^* = 3$, $x_3^* = 5$.

Properties

We can to obtain a recursive expression for the r-th moment of this model

$$E(X^r) = \sum_{j=1}^{k} I_j, \quad \alpha_k > r + 1,$$

where for $l = 1, \cdots, k-1$

$$I_l = \frac{C_{j-1}(\alpha_j - 1)}{x_{(j-1)}^*} \left[\left(log(x_j^*) - log(x_{j-1}^*)\right) 1(\alpha_j = r+1) + \frac{(x_j^*)^{r+1-\alpha_j} - (x_{j-1}^*)^{r+1-\alpha_j}}{r+1-\alpha_j} 1(\alpha_j \neq r+1)\right],$$

and

$$I_k = \frac{-(\alpha_k - 1) C_{k-1} (x_{k-1}^*)^r}{r + 1 - \alpha_k}.$$

Finally, we can generate samples through the inverse transform sampling for each piecewise

$$x = x_{(j-1)}^* \left(\frac{1-u}{C_{j-1}}\right)^{1/(1-\alpha_j)}, \quad (1 - C_{j-1}) \leq u \leq (1 - C_j),$$

where $u \sim Unif(0, 1)$.

Maximum likelihood estimator

The likelihood function related to the model is given by

$$L(\boldsymbol{\alpha}) = \prod_{j=1}^{k} \left[\prod_{i: x_i \in [x_{(j-1)}^*, x_{(j)}^*]} C_{j-1} \left(\frac{\alpha_j - 1}{x_{(j-1)}^*} \left(\frac{x_i}{x_{(j-1)}^*}\right)^{-\alpha_j}\right)\right].$$

The maximum likelihood estimators (MLEs) of the k parameters are

$$\hat{\alpha}_j = 1 + n_j \left(\sum_{i: x_i \in [x_{(j-1)}^*, x_{(j)}^*]} \log\left(\frac{x_i}{x_{(j-1)}^*}\right) + \sum_{h=j}^{k-1} n_{h+1} \log\left(\frac{x_{(j)}^*}{x_{(j-1)}^*}\right)\right)^{-1}, \quad j = 1, \ldots, k-1 \tag{3}$$

$$\hat{\alpha}_k = 1 + n_k \left(\sum_{i: x_i \geq x_{(k-1)}^*} \log\left(\frac{x_i}{x_{(k-1)}^*}\right)\right)^{-1}. \tag{4}$$

where $n_j = |i : x_i \in [x_{(j-1)}^*, x_{(j)}^*]|$.

Under regularity conditions, $\hat{\alpha} \xrightarrow{D} N(\alpha, I^{-1})$, where I^{-1} is the Fisher matrix with elements

$$I_{jj}^{-1} = -E\left[\frac{\partial^2 \log L(\alpha)}{\partial \alpha_j^2}\right] = \frac{(\alpha_j - 1)^2}{n_j}, \tag{5}$$

and $I_{jl}^{-1}(\alpha) = I_{jl}^{-1}(\alpha) = 0$, for $j \neq l$.

Bias correction for the maximum likelihood estimators

Following the procedure proposed by [1], the MLE with bias corrected are

$$\hat{\alpha}_j = 1 + (n_j - 1) \left(\sum_{i: x_i \in [x_{(j-1)}^*, x_{(j)}^*]} \log\left(\frac{x_i}{x_{(j-1)}^*}\right) + \sum_{h=1}^{k-1} n_{h+1} \log\left(\frac{x_{(h)}^*}{x_{(h-1)}^*}\right)\right)^{-1}, \quad j = 1, \ldots, k-1 \tag{6}$$

$$\hat{\alpha}_k = 1 + (n_k - 1) \left(\sum_{i: x_i \geq x_{(k-1)}^*} \log\left(\frac{x_i}{x_{(k-1)}^*}\right)\right)^{-1}. \tag{7}$$

Simulation study

We will present the performance of 10.000 replicates, using $x_1^* = 1.2$, $x_2^* = 2$, $x_3^* = 3$.

Table 1. Simulation study for three parameters, in parentheses are the results without the bias correction.

True values α_1 α_2 α_3	Estimators	$n = 100$ Bias	RMSE	CP	$n = 1000$ Bias	RMSE	CP
2.0 1.2 3.0	$\hat{\alpha}_1$	0.128 (0.127)	0.160 (0.160)	0.932 (0.945)	0.056 (0.057)	0.071 (0.071)	0.948 (0.950)
	$\hat{\alpha}_2$	0.061 (0.058)	0.075 (0.073)	0.827 (0.923)	0.026 (0.026)	0.033 (0.032)	0.930 (0.943)
	$\hat{\alpha}_3$	0.226 (0.226)	0.287 (0.291)	0.939 (0.948)	0.100 (0.098)	0.125 (0.124)	0.947 (0.954)
4.5 1.2 3.0	$\hat{\alpha}_1$	0.305 (0.304)	0.380 (0.382)	0.942 (0.950)	0.135 (0.137)	0.170 (0.173)	0.951 (0.946)
	$\hat{\alpha}_2$	0.120 (0.103)	0.143 (0.140)	0.632 (0.984)	0.052 (0.048)	0.064 (0.061)	0.861 (0.935)
	$\hat{\alpha}_3$	0.437 (0.472)	0.586 (0.667)	0.926 (0.960)	0.187 (0.190)	0.238 (0.243)	0.944 (0.952)
1.2 2.0 4.0	$\hat{\alpha}_1$	0.054 (0.051)	0.067 (0.064)	0.862 (0.922)	0.023 (0.023)	0.029 (0.029)	0.931 (0.940)
	$\hat{\alpha}_2$	0.120 (0.121)	0.151 (0.152)	0.929 (0.942)	0.053 (0.052)	0.066 (0.066)	0.948 (0.952)
	$\hat{\alpha}_3$	0.362 (0.366)	0.461 (0.470)	0.940 (0.952)	0.159 (0.161)	0.199 (0.203)	0.951 (0.950)
1.2 4.0 4.0	$\hat{\alpha}_1$	0.055 (0.052)	0.068 (0.065)	0.850 (0.927)	0.023 (0.023)	0.029 (0.029)	0.928 (0.945)
	$\hat{\alpha}_2$	0.272 (0.273)	0.341 (0.345)	0.944 (0.947)	0.121 (0.120)	0.151 (0.150)	0.951 (0.954)
	$\hat{\alpha}_3$	0.779 (0.849)	1.076 (1.306)	0.912 (0.956)	0.318 (0.331)	0.405 (0.423)	0.945 (0.952)
1.2 3.5 3.0	$\hat{\alpha}_1$	0.054 (0.052)	0.067 (0.065)	0.863 (0.923)	0.023 (0.023)	0.029 (0.029)	0.931 (0.943)
	$\hat{\alpha}_2$	0.229 (0.231)	0.290 (0.292)	0.946 (0.954)	0.103 (0.104)	0.130 (0.130)	0.948 (0.950)
	$\hat{\alpha}_3$	0.426 (0.449)	0.566 (0.626)	0.924 (0.953)	0.183 (0.180)	0.232 (0.231)	0.940 (0.951)
1.2 3.5 6.0	$\hat{\alpha}_1$	0.055 (0.051)	0.067 (0.065)	0.861 (0.925)	0.023 (0.023)	0.029 (0.028)	0.932 (0.941)
	$\hat{\alpha}_2$	0.235 (0.236)	0.294 (0.296)	0.940 (0.951)	0.102 (0.104)	0.128 (0.131)	0.952 (0.948)
	$\hat{\alpha}_3$	1.064 (1.123)	1.428 (1.558)	0.923 (0.952)	0.455 (0.453)	0.574 (0.582)	0.943 (0.950)

Estimating change points

Note that so far the change points are assumed to be fixed and known, which is not necessarily the case in practice. If we are interested in estimating it, we can use the procedure proposed in [2] using the *profile likelihood function*, which consists of replacing the MLE expressions of the parameters in the likelihood function and maximizing it in the change points, then we must insert the obtained values back to in (6) and (7).

Note: This procedure is not implemented as a simulation study so far in our manuscript, however, we can report that in some individual tests the procedure recovers the change points satisfactorily, and will be used in the future application section.

Maximum likelihood estimators under random censoring

In the presence random censoring the bias-corrected maximum likelihood estimators are

$$\hat{\alpha}_j = 1 + (d_j - 1) \left(\sum_{i: x_i \in [x_{(j-1)}^*, x_{(j)}^*]} \log\left(\frac{x_i}{x_{(j-1)}^*}\right) + \sum_{h=1}^{k-1} n_{h+1} \log\left(\frac{x_{(h)}^*}{x_{(h-1)}^*}\right)\right)^{-1}, \quad j = 1, \ldots, k-1 \tag{8}$$

$$\hat{\alpha}_k = 1 + (d_k - 1) \left(\sum_{i: x_i \geq x_{(k-1)}^*} \log\left(\frac{x_i}{x_{(k-1)}^*}\right)\right)^{-1}. \tag{9}$$

where $d_j = |i : x_i \in [x_{(j-1)}^*, x_{(j)}^*],\ x_i$ is a faillure time$|$. In this case, the elements of Fisher matrix are

$$I_{jj}^{-1}(\alpha) = -E\left[\frac{\partial^2}{\partial \alpha_j^2} l(\alpha \mid x^*, x, \delta)\right] = \frac{(\alpha_j - 1)^2}{d_j},$$

and $I_{jl}^{-1}(\alpha) = I_{jl}^{-1}(\alpha) = 0$, for $j \neq l$.

Simulation study with censored data

We will present the performance of the MLE under 50% of censored data using 10.000 replicates and the same configurations as the previous study.

Table 2. Simulation study for three parameters, in parentheses are the results without the bias correction.

True values α_1 α_2 α_3	Estimators	$n = 100$ Bias	RMSE	CP	$n = 1000$ Bias	RMSE	CP
2.0 1.2 3.0	$\hat{\alpha}_1$	0.138 (0.136)	0.173 (0.171)	0.928 (0.949)	0.043 (0.043)	0.054 (0.054)	0.947 (0.949)
	$\hat{\alpha}_2$	0.075 (0.067)	0.092 (0.085)	0.775 (0.918)	0.022 (0.021)	0.027 (0.027)	0.935 (0.951)
	$\hat{\alpha}_3$	0.378 (0.378)	0.480 (0.486)	0.918 (0.949)	0.117 (0.117)	0.146 (0.147)	0.946 (0.947)
4.5 1.2 3.0	$\hat{\alpha}_1$	0.372 (0.362)	0.459 (0.451)	0.912 (0.932)	0.115 (0.116)	0.144 (0.146)	0.951 (0.948)
	$\hat{\alpha}_2$	0.161 (0.138)	0.184 (0.214)	0.382 (0.999)	0.057 (0.055)	0.070 (0.068)	0.841 (0.925)
	$\hat{\alpha}_3$	2.233 (15.318)	3.490 (346.956)	0.224 (1.000)	0.831 (0.751)	1.012 (0.957)	0.743 (0.917)
1.2 2.0 4.0	$\hat{\alpha}_1$	0.061 (0.057)	0.075 (0.071)	0.814 (0.923)	0.018 (0.018)	0.022 (0.023)	0.942 (0.946)
	$\hat{\alpha}_2$	0.150 (0.149)	0.188 (0.189)	0.918 (0.943)	0.046 (0.046)	0.058 (0.058)	0.949 (0.951)
	$\hat{\alpha}_3$	0.648 (0.669)	0.828 (0.874)	0.906 (0.940)	0.199 (0.198)	0.249 (0.249)	0.948 (0.947)
1.2 4.0 4.0	$\hat{\alpha}_1$	0.065 (0.060)	0.080 (0.075)	0.810 (0.923)	0.019 (0.019)	0.024 (0.024)	0.936 (0.945)
	$\hat{\alpha}_2$	0.358 (0.350)	0.445 (0.441)	0.931 (0.945)	0.116 (0.114)	0.145 (0.143)	0.948 (0.950)
	$\hat{\alpha}_3$	2.511 (5.370)	5.085 (48.504)	0.488 (0.981)	0.677 (0.655)	0.849 (0.843)	0.891 (0.940)
1.2 3.5 3.0	$\hat{\alpha}_1$	0.063 (0.058)	0.077 (0.074)	0.810 (0.931)	0.018 (0.018)	0.023 (0.023)	0.942 (0.950)
	$\hat{\alpha}_2$	0.296 (0.299)	0.369 (0.377)	0.936 (0.947)	0.093 (0.094)	0.117 (0.117)	0.949 (0.950)
	$\hat{\alpha}_3$	1.175 (1.242)	1.535 (3.339)	0.641 (0.946)	0.319 (0.314)	0.399 (0.398)	0.921 (0.945)
1.2 3.5 6.0	$\hat{\alpha}_1$	0.064 (0.060)	0.079 (0.076)	0.823 (0.921)	0.019 (0.019)	0.024 (0.024)	0.934 (0.950)
	$\hat{\alpha}_2$	0.309 (0.307)	0.386 (0.388)	0.935 (0.952)	0.100 (0.099)	0.125 (0.124)	0.946 (0.952)
	$\hat{\alpha}_3$	3.011 (7.168)	4.249 (224.612)	0.652 (0.940)	0.776 (0.765)	0.971 (0.976)	0.924 (0.947)

Empirical application

We consider the data organized by [4], which was obtained from De Imperatoribus Romanis.

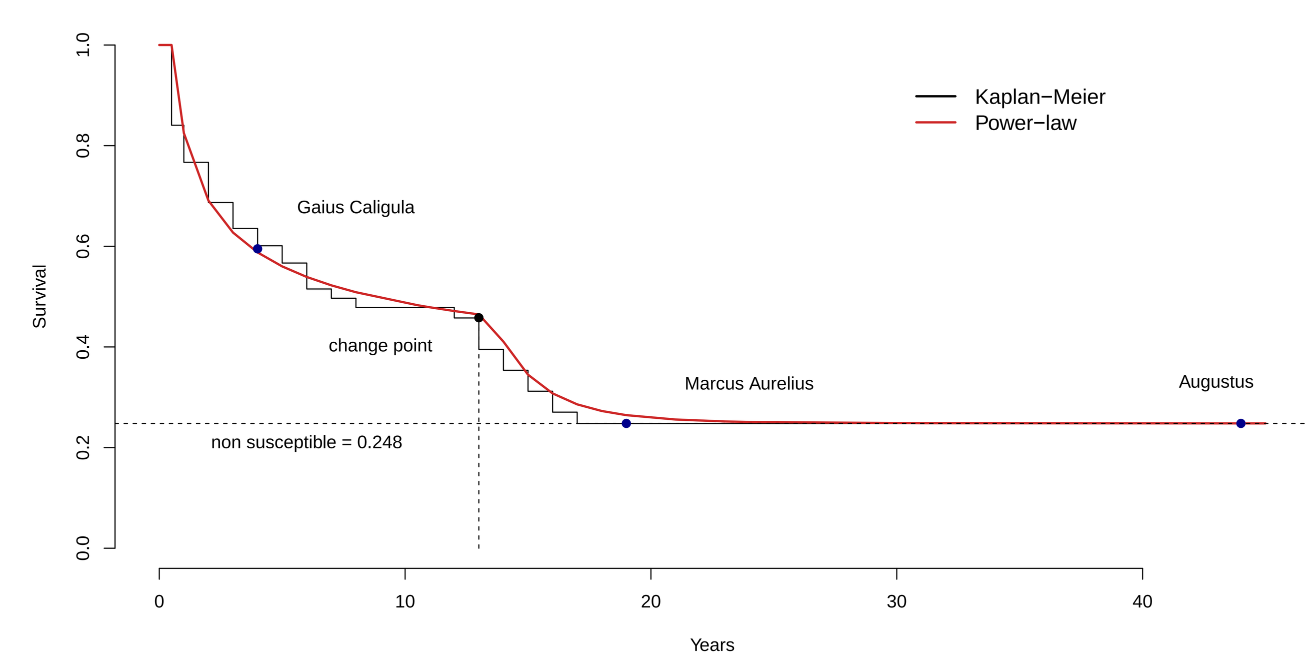

Figure 2. Power-law survivor function of Roman emperors. Source: [3]

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