On the role of U-matrices in the commutativity condition defining a special class of OBS

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Abbreviated abstract: Models with orthogonal block structure (OBS) are an important class of linear mixed models, based on the structure of the covariance matrix. Imposing a commutativity condition, we get models with commutative orthogonal block structure (COBS), obtaining least squares estimators giving best linear unbiased estimators for estimable vectors. Using U matrices, we present an alternative commutativity condition to the one used when Fonseca (2008) introduced COBS.

Related publications:

Fonseca, M.; Mexia, J. T.; Zmyslony, R. (2008) Inference in normal models with commutative orthogonal block structure. *Acta et Commentationes Universitatis Tartuensis de Mathematica*12, 3–16.

Santos, C.; Nunes, C.; Dias, C.; Mexia J.T. (2020) Models with commutative orthogonal block structure: a general condition for commutativity. *Journal of Applied Statistics.* 2421-2430, DOI: 10.1080/02664763.2020.1765322

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Models with orthogonal block structure

Mixed model

$$\boldsymbol{Y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i$$

$\boldsymbol{\beta}_0$ is fixed

$\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_w$ are independent random vectors w/ null mean vectors, covariance matrices $\sigma_1^2 \boldsymbol{I}_{c_1} \dots \sigma_w^2 \boldsymbol{I}_{c_w}$, $c_i = rank(\boldsymbol{X_i})$, $i = 1, \dots, w$ and null cross-covariance matrices.

Mean vector: $\boldsymbol{\mu} = \boldsymbol{X}_0 \boldsymbol{\beta}_0$

Covariance matrix: $\boldsymbol{V}(\boldsymbol{\theta}) = \sum_{i=1}^{w} \sigma_i^2 \boldsymbol{M}_i$ where $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i^T, i = 1, \dots, w$.

Orthogonal projection matrix on the space spanned by the mean vector

$$\boldsymbol{T} = \boldsymbol{X}_0 (\boldsymbol{X}_0^T \boldsymbol{X}_0)^+ \boldsymbol{X}_0^T = \boldsymbol{X}_0 \boldsymbol{X}_0^+$$

- When $\boldsymbol{M}_1, \dots, \boldsymbol{M}_w$ commute, they generate a commutative Jordan algebra of symmetric matrices (CJAS) A.
- The principal basis, $pb(A) = Q$, constituted by known pairwise orthogonal orthogonal projection matrices.
- $\boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j} \boldsymbol{Q}_j$ (matrices $\boldsymbol{M}_i$, $i = 1, \dots, w$ are linear combinations of the matrices of $pb(A)$.
- Covariance matrix of $\boldsymbol{Y}$: $\boldsymbol{V} = \sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j$
  with $\gamma_j = \sum_{i=1}^{w} b_{i,j} \sigma_i^2, j = 1, \dots, m$, the canonical variance components.

Since $\sum_{i=1}^{w} \boldsymbol{M}_j \in$ A is invertible, A is a complete CJAS and

$$\sum_{j=1}^{m} \boldsymbol{Q}_j = \boldsymbol{I}_n$$

$\boldsymbol{Y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i$ is a model with orthogonal block structure, OBS (Nelder, 1965a,1965b)

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Models with commutative orthogonal block structure

OBS

COBS

$\boldsymbol{T}$, the OPM on the space spanned by the mean vector, commutes with the POOPM $\boldsymbol{Q}_j, j = 1, \dots, m$.

the OPM $\boldsymbol{T}$ also commutes with the covariance matrix of Y, $\boldsymbol{V}$.

A general condition for the **commutativity** between $\boldsymbol{T}$ and $\boldsymbol{V}$ can be achieved using a partition of the covariance matrix and resorting to U-matrices.

Assuming the rows of matrix $\boldsymbol{X_0}$ to correspond to the sets of levels of the fixed effects factors, the mean values of the observations will be determined by those sets.

Let us consider $\dot{n}$ sets of levels associated to $r_1, \dots, r_{\dot{n}}$, contiguous rows of $\boldsymbol{X}_0$. If the components of $\boldsymbol{\beta}_0$ are the corresponding mean values, we can reorder the observations to have the block diagonal matrix $\boldsymbol{X}_0 = D(\mathbf{1}_{r_1}, \dots, \mathbf{1}_{r_{\dot{n}}})$, where $\mathbf{1}_{r_l}$ is a vector with all r_l, $l = 1, \dots, \dot{n}$, components equal to 1.

The orthogonal projection matrix on the space spanned by the mean vector, will be $\boldsymbol{T} = D\left(\frac{1}{r_1}\boldsymbol{J}_{r_1}, \dots, \frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}}\right)$, where $\boldsymbol{J}_{r_l} = \mathbf{1}_{r_l}\mathbf{1}_{r_l}^T$, $l = 1, \dots, \dot{n}$.

The fundamental partition of $\boldsymbol{Y}$ will be constituted by the sub-vectors $\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_{\dot{n}}$, corresponding to the $\dot{n}$ sets of the levels of the fixed effects factors.

Then the covariance matrix can be defined by

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_{1,1} & \cdots & \boldsymbol{V}_{1,\dot{n}} \\ \vdots & & \vdots \\ \boldsymbol{V}_{\dot{n},1} & \cdots & \boldsymbol{V}_{\dot{n},\dot{n}} \end{bmatrix},$$

with $\boldsymbol{V}_{l,l}$ the covariance matrix of $\boldsymbol{Y}_l$, $l = 1, \dots, \dot{n}$, and $\boldsymbol{V}_{l,h}$ the cross-covariance matrix of $\boldsymbol{Y}_l$ and $\boldsymbol{Y}_h$, $l \neq h$.

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Results and Conclusions

In COBS, $\boldsymbol{T}$ and $\boldsymbol{V}$ commute

$$\begin{bmatrix} \frac{1}{r_1}\boldsymbol{J}_{r_1}\boldsymbol{V}_{1,1} & \cdots & \frac{1}{r_1}\boldsymbol{J}_{r_1}\boldsymbol{V}_{1,\dot{n}} \\ \vdots & & \vdots \\ \frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}}\boldsymbol{V}_{\dot{n},1} & \cdots & \frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}}\boldsymbol{V}_{\dot{n},\dot{n}} \end{bmatrix} = \boldsymbol{TV} = \boldsymbol{VT} = \begin{bmatrix} \boldsymbol{V}_{1,1}\frac{1}{r_1}\boldsymbol{J}_{r_1} & \cdots & \boldsymbol{V}_{1,\dot{n}}\frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}} \\ \vdots & & \vdots \\ \boldsymbol{V}_{\dot{n},1}\frac{1}{r_1}\boldsymbol{J}_{r_1} & \cdots & \boldsymbol{V}_{\dot{n},\dot{n}}\frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}} \end{bmatrix}$$

if and only if
$$\begin{cases} \frac{1}{r_1}\boldsymbol{J}_{r_1}\boldsymbol{V}_{1,1} = \boldsymbol{V}_{1,1}\frac{1}{r_1}\boldsymbol{J}_{r_1} & \cdots & \frac{1}{r_1}\boldsymbol{J}_{r_1}\boldsymbol{V}_{1,\dot{n}} = \boldsymbol{V}_{1,\dot{n}}\frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}} \\ \vdots & & \vdots \\ \frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}}\boldsymbol{V}_{\dot{n},1} = \boldsymbol{V}_{\dot{n},1}\frac{1}{r_1}\boldsymbol{J}_{r_1} & \cdots & \frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}}\boldsymbol{V}_{\dot{n},\dot{n}} = \boldsymbol{V}_{\dot{n},\dot{n}}\frac{1}{r_{\dot{n}}}\boldsymbol{J}_{r_{\dot{n}}} \end{cases}$$

Thes

$\boldsymbol{V}$ and $\boldsymbol{T}$ commute if and only if the sub-matrices $\boldsymbol{V}_{l,\boldsymbol{h}}$, $l, h = 1, \ldots, m$, are U-matrices.

Since, in COBS, $\boldsymbol{T}$ and $\boldsymbol{V}$ commute, the least squares estimators, LSE, for estimable vectors, will be BLUE (best linear unbiased estimators) whatever the variance components. (Zmyślony, 1980)

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